Orthogonal Friends

Parth Patel

Introduction:

Most calculus students at some point in their education will study differential equations and their many applications, including modeling many natural phenomena.

We explore a certain application of the differential equation, namely finding the **orthogonal trajectories** of a curve. The curves explored may not fit the formal definition for an orthogonal trajectory, so we will define some new terms to fit our needs.

This study relies heavily on the use of software to depict the graphs of various intricate functions and is experienced best when read with an open copy of a Desmos file used to create the provided images throughout the paper. The sliders can be used to change the variables as desired. The file is linked here: https://rb.gy/fn72i.

Throughout this paper I will address the following topics:

- the traditional solutions of standard "Orthogonal Trajectory Problems";
- different variations on the classical orthogonal trajectory;
- why those variations are not *true* orthogonal trajectories; and
- further insights on the intersections of graphs.

Traditional Orthogonal Trajectories:

Let's begin with a definition.

Definition: Orthogonal Trajectory

In mathematics, an *orthogonal trajectory* is a curve which intersects any curve of a given pencil of planar curves orthogonally. Here a *pencil* is a set of functions defined by a parameter such as k, and *orthogonal* simply means "at a right angle".

Here, we can consider the tangent lines to the functions at the point of intersection. They should be perpendicular.

These orthogonal trajectories are the traditional solutions to the problems from a first-year calculus course and in many other classrooms. The question may read:

Find the orthogonal trajectories of the family of functions $y = kx^2$.

We know that the intersection should occur at a right angle, so the first derivatives of the functions should be negative reciprocals:

$$\frac{dy}{dx} = 2kx \text{ thus } \frac{dy}{dx} = \frac{-1}{2kx}.$$

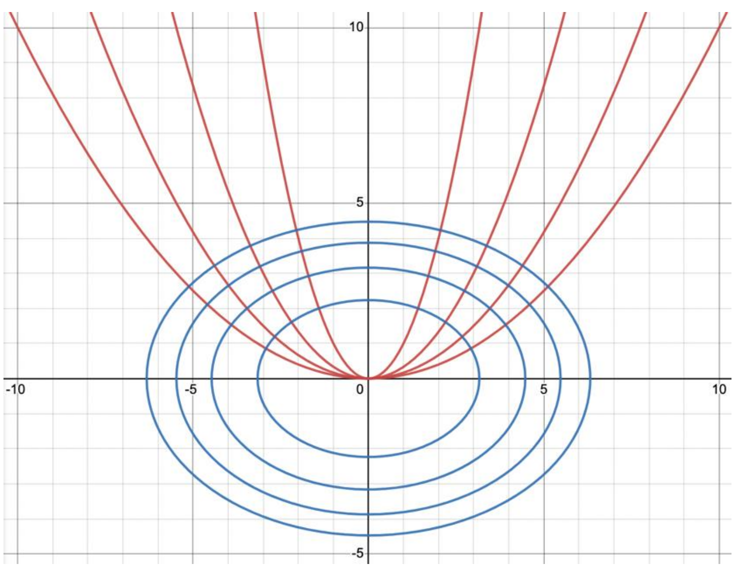
We also know that $k = \frac{y}{x^2}$, so we can substitute that and rearrange our differential equation to prepare for integration:

$$\frac{dy}{dx} = \frac{-1}{2x\frac{y}{x^2}} \Leftrightarrow 2ydy = -xdx \Leftrightarrow \int 2ydy = \int -xdx.$$

After integrating (making sure to add that pesky constant) and rearranging, we have the solution.

$$2y^2 + x^2 = c.$$

So, we say that the orthogonal trajectory of $y = kx^2$ is $2y^2 + x^2 = c$.



A picture of several functions of the pencil $y = kx^2$ (red) and its orthogonal trajectory, $2y^2 + x^2 = c$ (blue). Note that each member of one pencil intersects *every* member of the other pencil orthogonally.

This is the standard solution of the problem seen in most calculus texts. However, on the day my teacher planned to introduce orthogonal trajectories, she ran out of class time, so she told us to try the homework and that we'd have the lesson in class the next day. As instructed, I did the homework. Here was my solution, with the same start:

$$\frac{dy}{dx} = 2kx \text{ thus } \frac{dy}{dx} = \frac{-1}{2kx}$$

Let's separate the differential equation:

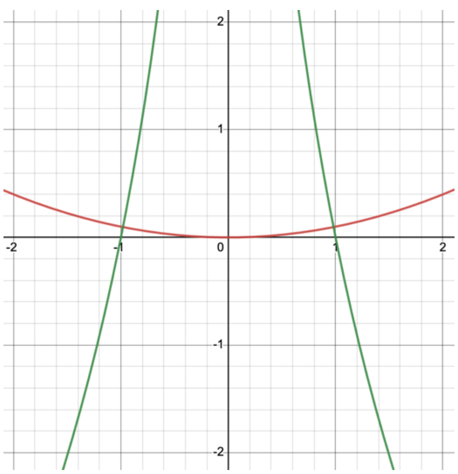
$$dy = -\frac{dx}{2kx} \Leftrightarrow \int dy = -\int \frac{dx}{2kx}.$$

No substitution is necessary to obtain the solution

$$y = \frac{-\ln|x|}{2k} + c.$$

The knee-jerk response may be that it is wrong because it is not a *true* orthogonal trajectory of $y = kx^2$. However, it still turns out to be interesting.

The graphs are orthogonal, and that night, as I checked my calculus homework, I thought I had correctly solved the problem. But my solution has the connection between $y = kx^2$ and $y = \frac{-\ln|x|}{2k} + c$. The parameter k remains in both equations.



A picture of $y = \frac{-\ln|x|}{2k} + c$ (green) and $y = kx^2$ (red) with $k = \frac{1}{10}$ and $c = 0$. The graphs remain mutually orthogonal for all $k, c \in \mathbb{R}$, as can be seen in the Desmos project.

I was intrigued. This is not an orthogonal trajectory (recall that a member of one pencil must intersect *every* member of the other pencil orthogonally). But it's too special not to merit a definition.

> **Definition: Orthogonal Friend**
>
> Two graphs are *orthogonal friends* if they are related by some parameter k and intersect orthogonally for all $k \in \mathbb{R}$ for which both functions are defined.

To summarize so far, now we know that $y = kx^2$ has both a proper orthogonal trajectory and another related graph which is not truly an orthogonal trajectory but is maybe close enough for us to affectionately label it an orthogonal friend.

Remember that the orthogonal friend was obtained by *not* substituting for the parameter k during the integration step. So, the orthogonal friend still contains a parameter:

$$y = \frac{-\ln|x|}{2k} + c.$$

Let's try substituting for k now and see what happens:

$$y = \frac{-\ln|x|}{2\frac{y}{x^2}} + c = \frac{-x^2\ln|x|}{2y} + c \iff y^2 = \frac{-x^2\ln|x|}{2} + cy.$$

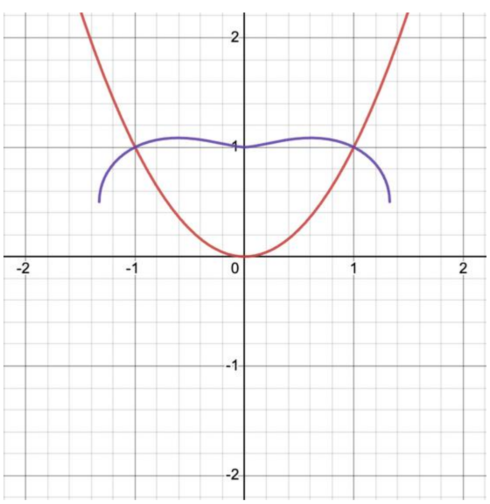
Let's rewrite the equation as

$$y^2 - cy + \frac{x^2\ln|x|}{2} = 0.$$

And now we can solve for y:

$$y = \frac{c \pm \sqrt{c^2 - 2x^2\ln|x|}}{2}.$$

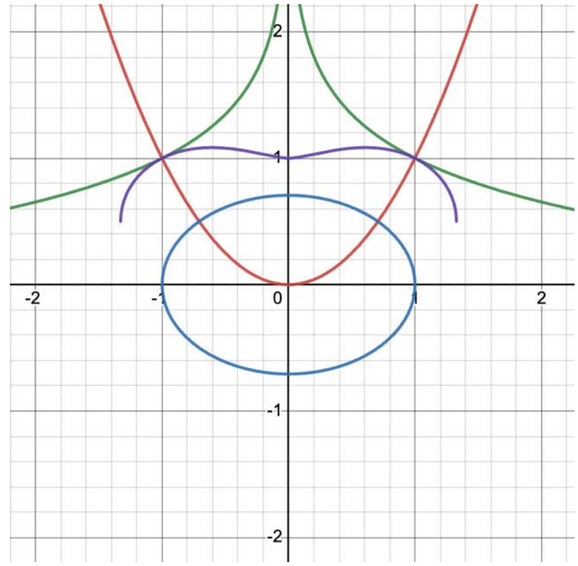
Let's take the positive root because $x^2 \epsilon\ \mathbb{R}^+$. There are now two variables to manipulate, both c and k. Take a look at the graph for $c = k$.



A picture of $y = \frac{c + \sqrt{c^2 - 2x^2\ln|x|}}{2}$ (purple). It's orthogonal to $y = x^2$ (red) as well, but there's a catch. The graphs are orthogonal only when $c = k$, as can be seen using the Desmos project.

What a beautiful result! Not only have we produced the traditional orthogonal trajectory, but we have also produced an additional graph, what we now call our *orthogonal friend*, and we derived a different equation from it. Let's enjoy all the graphs together:

- The original function: $y = x^2$ (red),
- The orthogonal trajectory: $2y^2 + x^2 = c$ (blue),
- The orthogonal friend: $y = \frac{-\ln|x|}{2k} + c$ (green),
- This new function: $y = \frac{c + \sqrt{c^2 - 2x^2 \ln|x|}}{2}$ (purple).



Consider another example, $y = kx^3$, for which

$$\frac{dy}{dx} = 3kx^2.$$

We will generate two differential equations, both by taking the reciprocal of the right-hand side, but for second, substitute $k = \frac{y}{x^3}$:

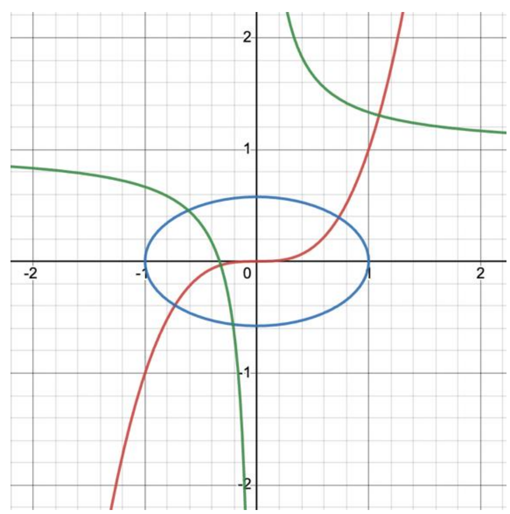
$$dy = \frac{-dx}{3kx^2} \quad \text{and} \quad 3ydy = -xdx.$$

Let's integrate to find the equations:

$$y = \frac{1}{3kx} + c \quad \text{and} \quad x^2 + 3y^2 = c.$$

Graphing them in Desmos, we see the following:

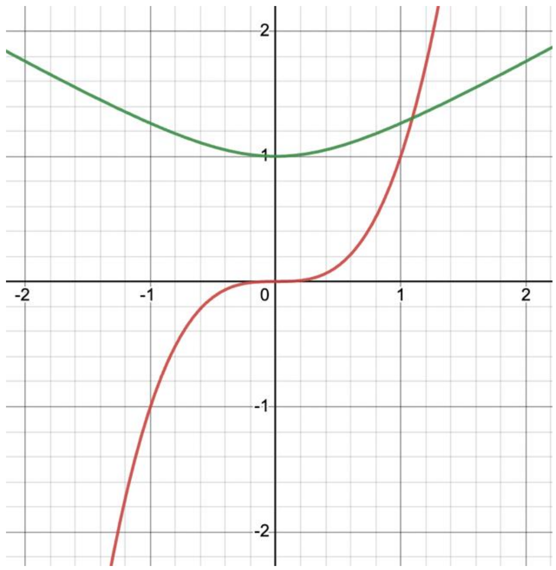
- The original function: $y = kx^3$ (red),
- The orthogonal trajectory: $x^2 + 3y^2 = c$ (blue), and
- The orthogonal friend: $y = \frac{1}{3kx} + c$ (green).



They all are orthogonal to $y = x^3$. Let's now find the final graph by substituting $k = \frac{y}{x^3}$ into $y = \frac{1}{3kx} + c$.

After substitution, we find that $y = \frac{x^2}{3y} + c$ so $y^2 - cy - \frac{x^2}{3} = 0$. Since the original function has a range $\mathbb{R}$, let's leave this final equation in implicit form.

Graphing, we get the following:



The graphs are not orthogonal at all, and, in fact, no value of *c* makes the graphs orthogonal. Substituting *k* after integration fails. Perhaps for $y = kx^2$, we simply had a coincidence.

We can try to generalize this result for any $y = x^n$ for $n \in \mathbb{N}$. And while we could put a coefficient on the *x* term, working without one will make the entire process easier to understand. Let's see if we can derive any expressions for various orthogonal graphs of powers of *x*, which should include one true orthogonal trajectory. Differentiating $y = x^n$, we find

$$\frac{dy}{dx} = nx^{n-1}.$$

Taking the negative reciprocal, we will find two solutions, both of which will be dependent on n:

$$\frac{dy}{dx} = \frac{-1}{nx^{n-1}} \Leftrightarrow y\frac{dy}{dx} = \frac{-1}{n\frac{x^{n-1}}{y}}.$$

Substituting $\frac{1}{x} = \frac{x^{n-1}}{y}$, we get

$$\int nydy = \int -xdx.$$

Hence,

$$\frac{ny^2}{2} = \frac{-x^2}{2} + c.$$

So, the solution is the ellipse given by $ny^2 + x^2 = C$. No surprise there.

Now, we restart:

$$\frac{dy}{dx} = nx^{n-1}.$$

Taking the negative reciprocal to find the new solution, we have

$$\frac{dy}{dx} = \frac{-1}{nx^{n-1}}.$$

Thus,

$$\int dy = \int \frac{-dx}{nx^{n-1}}.$$

Hence, as long as $n \neq 2$,

$$y = \frac{1}{n(n-2)x^{n-2}} + C.$$

The derivations are complete, and our results are as follows:

- The original function: $y = kx^n$.
- The standard orthogonal trajectory: $ny^2 + x^2 = C$.
- The new and interesting orthogonal friend: $y = \frac{1}{n(n-2)x^{n-2}} + C$.

Recall our study of $y = x^2$, specifically when we derived the formula for its orthogonal friend and then substituted $k = \frac{y}{x^2}$ into the equation. The resulting function was orthogonal to $y = x^2$ only when the integration constant *c* was equal to *k* but had quickly realized that it was a coincidence when the same procedure failed for $y = x^3$. Let's see if we can derive another orthogonal friend for $y = x^n$ from the results of $y = x^2$.

The orthogonal friend of $y = x^2$ is $y = \frac{-\ln|x|}{2k} + c$. Instead of $k = \frac{y}{x^2}$, Let's substitute $k = \frac{y}{x^n}$ and see what happens. We find the following:

$$y = \frac{-x^n \ln|x|}{2y} + c.$$

We also need the graph to contain (1,1), so $c = 1$. Also, when we take the derivative of $y = x^n$ and then the negative reciprocal, we will end up with an *n* in the denominator. So, while it may seem strange now, substitute n in the denominator for $2y$. We now have

$$y = 1 - \frac{x^n \ln|x|}{n};$$

. $|x|$ allows the largest possible domain. We now have

$$y = 1 - \frac{|x|^n \ln|x|}{n}.$$

So, just like $y = x^n$, this new graph always contains (1,1). Let's now set the derivative of our new function equal to the negative reciprocal of the derivative of $y = x^n$, with careful use of the product rule:

$$\frac{-1}{nx^{n-1}} = \frac{-n|x|^{n-1}\ln|x|}{n} - \frac{x^{n-1}}{n}.$$

With some cancellation and rearrangement, we find:

$$\frac{1}{x^{n-1}} = n|x|^{n-1}\ln|x| - x^{n-1}.$$

Plugging in our point of intersection, (1,1):

$$\frac{1}{1^{n-1}} = n * 1^{n-1}\ln|1| + 1^{n-1},$$

which is true for any $n \in \mathbb{N}$, so our two graphs contain (1,1). Furthermore, their derivatives will always be negative reciprocals at that point, and thus the two functions are orthogonal friends. The exponent of x can be any positive real number, c. So, we have shown that $y = 1 - \frac{x^c \ln|x|}{n}$ is a pencil of orthogonal friends of $y = x^n$.

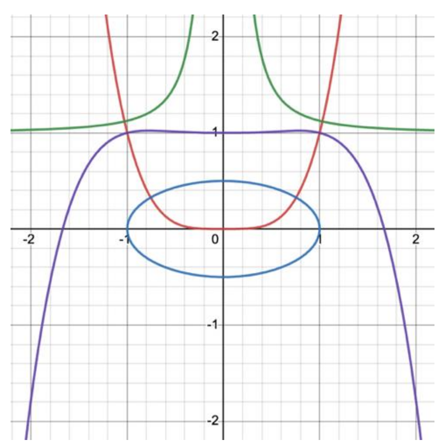
Definition: Orthogonal Friend Group

A graph has an *orthogonal friend group* if there is a pencil of functions defined by parameters, with at least one unique parameter, such that each member of that pencil is an orthogonal friend of the graph.

Since $y = 1 - \frac{x^c \ln|x|}{n}$ contains a unique parameter c, it is an orthogonal friend group of $y = x^n$.

To recap, we explicated the function, and our results are as follows:

- The original function: $y = x^n$ (red)
- The standard orthogonal trajectory: $ny^2 + x^2 = C$ (blue).
- The new and interesting orthogonal friend: $y = \frac{1}{n(n-2)x^{n-2}} + C$ (green).
- The new and even more interesting orthogonal friend group:
- $y = 1 - \frac{x^c \ln|x|}{n}$ (purple).



Here is a picture of the function and its orthogonal trajectory, its orthogonal friend, and a single member of its orthogonal friend group.